

Enumerations of $(K_4 - e)$ -designs with small orders ¹

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Dedicated to the memory of Lucia Gionfriddo (1973-2008)

Abstract: It is established that up to isomorphism, there are only one $(K_4 - e)$ -design of order 6, three $(K_4 - e)$ -designs of order 10 and two $(K_4 - e)$ -designs of order 11. As an application of our enumerative results, we discuss the fine triangle intersection problem for $(K_4 - e)$ -designs of orders $v = 6, 10, 11$.

Keywords: $(K_4 - e)$ -design; enumeration; intersection; fine triangle intersection

1 Introduction

Let H be a simple graph and G be a subgraph of H . A G -design of H (or (H, G) -design) is a pair (X, \mathcal{B}) where X is the vertex set of H and \mathcal{B} is an edge-disjoint decomposition of H into isomorphic copies (called *blocks*) of the graph G . If H is the complete graph K_v , we refer to such a G -design as one of order v .

The most basic question in design theory is that given a graph G and a positive integer v , whether a G -design of order v exists. If the existence problem is answered completely, then a further question is what about the enumeration problem for G -designs of order v . That is to say up to isomorphism, how many G -designs of order v exist? Two G -designs of order v (X, \mathcal{B}_1) and (X, \mathcal{B}_2) are said to be *isomorphic* if there exists a permutation π on X such that $\pi(\mathcal{B}_1) = \mathcal{B}_2$, where π is applied to the elements of each block of \mathcal{B}_1 . For more information on G -designs, the interested reader may refer to [1].

If G is the graph with vertices a, b, c, d and edges ab, ac, ad, bc, bd , then such a graph is called a $K_4 - e$ and denoted by $[a, b, c - d]$. Bermond and Schönheim

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established that [2] a $(K_4 - e)$ -design of order v exists if and only if $v \equiv 0, 1 \pmod{5}$ and $v \geq 6$. In this paper we will focus on the enumerations of $(K_4 - e)$ -designs of order v for $v = 6, 10, 11$. We shall show that there is only one $(K_4 - e)$ -design of order 6 up to isomorphism. There are three non-isomorphic $(K_4 - e)$ -designs of order 10 and two non-isomorphic $(K_4 - e)$ -designs of order 11.

Finally, as an application of our enumerative results, we investigate the fine triangle intersection problem for $(K_4 - e)$ -designs of orders $v = 6, 10, 11$.

2 Enumerations of $v = 6$ and 10

Theorem 2.1 *There is only one $(K_4 - e)$ -design of order 6 up to isomorphism.*

Proof Let $X = \{0, 1, 2, 3, 4, 5\}$. Suppose that (X, \mathcal{B}) is any $(K_4 - e)$ -design of order 6. For every $x \in X$, denote by $d_i(x)$, $i = 2, 3$, the number of blocks of \mathcal{B} in which the degree of x is i . It follows that $2d_2(x) + 3d_3(x) = 5$, and so $d_2(x) = d_3(x) = 1$. It is readily checked that up to isomorphism the unique $(K_4 - e)$ -design of order 6 is:

$$\mathcal{B} : [0, 1, 2 - 3], [2, 3, 4 - 5], [4, 5, 0 - 1]. \quad \square$$

Lemma 2.2 *Any $(K_4 - e)$ -design of order 10 contains a subdesign of order 6.*

Proof Let (X, \mathcal{B}) be any $(K_4 - e)$ -design of order 10. For every $x \in X$, denote by $d_i(x)$, $i = 2, 3$, the number of blocks of \mathcal{B} in which the degree of x is i . It follows that $2d_2(x) + 3d_3(x) = 9$. Solving this equation gives two possibilities: $d_2(x) = 0$ and $d_3(x) = 3$ (we refer to such a vertex as a a -element) or $d_2(x) = 3$ and $d_3(x) = 1$ (we refer to such a vertex as a b -element). Denote the number of a -elements and b -elements by α and β , respectively. Since each block contains exactly two elements with degree 3 we have

$$\begin{cases} 3\alpha + \beta = 18 \\ \alpha + \beta = 10 \end{cases}$$

and so $\alpha = 4$ and $\beta = 6$. Let $A = \{6, 7, 8, 9\}$ and $B = \{0, 1, 2, 3, 4, 5\}$ be the sets of a -elements and b -elements, respectively. Then we have the following six blocks $\mathcal{C} = \{[6, 7, x_1 - x_2], [8, 9, x_1 - x_2], [6, 8, x_3 - x_4], [7, 9, x_3 - x_4], [6, 9, x_5 - x_6], [7, 8, x_5 - x_6]\}$, where $\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{0, 1, 2, 3, 4, 5\}$, and so $(B, \mathcal{B} \setminus \mathcal{C})$ is a $(K_4 - e)$ -design of order 6. \square

Theorem 2.3 *There are exactly 3 non-isomorphic $(K_4 - e)$ -designs of order 10.*

Proof Let $X = \{0, 1, 2, \dots, 9\}$. Suppose that (X, \mathcal{B}) is any $(K_4 - e)$ -design of order 10. Note that there is the only unique $(K_4 - e)$ -design of order 6 under isomorphism. By Lemma 2.2, (X, \mathcal{B}) contains a subdesign of order 6, say, $[0, 1, 2 - 3], [2, 3, 4 - 5], [4, 5, 0 - 1]$. The other 6 blocks of \mathcal{B} must be the following forms: $[6, 7, x_1 - x_2]$,

$[8, 9, x_1 - x_2], [6, 8, x_3 - x_4], [7, 9, x_3 - x_4], [6, 9, x_5 - x_6], [7, 8, x_5 - x_6]\}$, where $\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{0, 1, 2, 3, 4, 5\}$.

Let $D = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$ and $N = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$. We distinguish the following three cases.

(1) If $|D \cap N| = 3$, without loss of generality we can always assume that $\{x_1, x_2\} = \{0, 1\}$. We then have $\{x_3, x_4\} = \{2, 3\}$ and $\{x_5, x_6\} = \{4, 5\}$ under isomorphism.

(2) If $|D \cap N| = 1$, similarly we can always assume that $\{x_1, x_2\} = \{0, 1\}$. We then have $\{x_3, x_4\} = \{2, 4\}$ and $\{x_5, x_6\} = \{3, 5\}$ under isomorphism.

(3) If $|D \cap N| = 0$, similarly we can always assume that $\{x_1, x_2\} = \{0, 2\}$. We then have $\{x_3, x_4\} = \{1, 4\}$ and $\{x_5, x_6\} = \{3, 5\}$ under isomorphism.

From the above discussions, we have the following three $(K_4 - e)$ -designs of order 10 under isomorphism:

$$\mathcal{B}_1 : \begin{array}{ccccc} [0, 1, 2 - 3], & [2, 3, 4 - 5], & [4, 5, 0 - 1], & [6, 7, 0 - 1], & [8, 9, 0 - 1], \\ [6, 8, 2 - 3], & [7, 9, 2 - 3], & [6, 9, 4 - 5], & [7, 8, 4 - 5]; \end{array}$$

$$\mathcal{B}_2 : \begin{array}{ccccc} [0, 1, 2 - 3], & [2, 3, 4 - 5], & [4, 5, 0 - 1], & [6, 7, 0 - 1], & [8, 9, 0 - 1], \\ [6, 8, 2 - 4], & [7, 9, 2 - 4], & [6, 9, 3 - 5], & [7, 8, 3 - 5]; \end{array}$$

$$\mathcal{B}_3 : \begin{array}{ccccc} [0, 1, 2 - 3], & [2, 3, 4 - 5], & [4, 5, 0 - 1], & [6, 7, 0 - 2], & [8, 9, 0 - 2], \\ [6, 8, 1 - 4], & [7, 9, 1 - 4], & [6, 9, 3 - 5], & [7, 8, 3 - 5]. \end{array}$$

It is easy to see that any two of the three $(K_4 - e)$ -designs of order 10 are not isomorphic. This completes the proof. \square

3 Enumeration of $v = 11$

Lemma 3.1 *Let (X, \mathcal{B}) be any $(K_4 - e)$ -design of order 11. Let $D = \{\{x, y\} : [x, y, z - u] \in \mathcal{B}\}$. Then (X, D) is a cycle of length 11.*

Proof For every $x \in X$, denote by $d_i(x)$, $i = 2, 3$, the number of blocks of \mathcal{B} in which the degree of x is i . It follows that $2d_2(x) + 3d_3(x) = 10$, which gives two solutions: $d_2(x) = 2$ and $d_3(x) = 2$ (we refer to such a vertex as a a -element) or $d_2(x) = 5$ and $d_3(x) = 0$ (we refer to such a vertex as a b -element). Denote the number of a -elements and b -elements by α and β , respectively. Since each block contains exactly two elements with degree 3 we have $2\alpha = 22$ and $\alpha + \beta = 11$, and so $\alpha = 11$ and $\beta = 0$. We then have the fact that for every $x \in X$

$$d_2(x) = d_3(x) = 2. \tag{1}$$

By the definition of D , (X, D) is a 2-regular graph, whose connected component is a cycle with length at least 3. Let $X = \{0, 1, 2, \dots, 10\}$. We will show that (X, D) is a cycle of length 11. All possibilities are exhausted as below.

(1). If (X, D) contains a cycle of length 4, without loss of generality, we can assume that \mathcal{B} contain the following four blocks: $[0, 1, x_1, x_2]$, $[1, 2, x_3, x_4]$, $[2, 3, x_5, x_6]$, $[3, 0, x_7, x_8]$, where $x_1, x_2, \dots, x_8 \in X$. From the fact (1), $\{0, 2\}$ must appear in some one of the four above-listed blocks. Since (X, \mathcal{B}) is a $(K_4 - e)$ -design of order 11, we have $x_i \notin \{0, 1, 2, 3\}$ for each $1 \leq i \leq 8$. It is impossible.

(2). If (X, D) contains a cycle of length 5, similarly we can assume that \mathcal{B} contain the following five blocks: $[0, 1, x_1, x_2]$, $[1, 2, x_3, x_4]$, $[2, 3, x_5, x_6]$, $[3, 4, x_7, x_8]$, $[4, 0, x_9, x_{10}]$, where $x_1, x_2, \dots, x_{10} \in X$. From the fact (1), the 5 remaining 2-subsets of $\{0, 1, \dots, 4\}$, $\{0, 2\}$, $\{0, 3\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, must appear in some one of the five above-listed blocks. Note that $|\{x_{2i-1}, x_{2i}\} \cap \{0, 1, \dots, 4\}| \leq 1$ for each $i = 1, 2, \dots, 5$. If $|\{x_{2i-1}, x_{2i}\} \cap \{0, 1, \dots, 4\}| = 1$ for some i , then there appear exactly two 2-subsets of $\{\{0, 2\}, \{0, 3\}, \{1, 3\}, \{1, 4\}, \{2, 4\}\}$ in the block containing $\{x_{2i-1}, x_{2i}\}$. A contradiction occurs.

(3). If (X, D) contains a cycle of length 3, similarly we can assume that \mathcal{B} contain the following three blocks: $[0, 1, 3 - 4]$, $[1, 2, 5 - 6]$, $[2, 0, 7 - 8]$. Since $\{0, x\}$, $\{1, x\}$ and $\{2, x\}$ where $x \in \{9, 10\}$ must appear in some block of \mathcal{B} , by the fact (1) we can further assume that \mathcal{B} contain the following blocks: $[9, 10, 0 - 1]$, $[9, c, 2 - *]$, $[10, d, 2 - *]$, $[e, f, 0 - *]$, $[g, h, 1 - *]$ under isomorphism. Since the unused 2-subsets containing 2 are only $\{2, 3\}$ and $\{2, 4\}$. So, we have $c = 3$ and $d = 4$, or $c = 4$ and $d = 3$. Under isomorphism, we can assume that $c = 3$ and $d = 4$. Similar arguments give $\{e, f\} = \{5, 6\}$ and $\{g, h\} = \{7, 8\}$. From the fact (1), other three blocks of \mathcal{B} can be assumed in the two possibilities under isomorphism:

Case I: $[3, 4, * - *]$, $[5, 7, * - *]$, $[6, 8, * - *]$;

Case II: $[3, 5, * - *]$, $[4, 7, * - *]$, $[6, 8, * - *]$.

For Case I, (X, D) contains a cycle of length 4: $\{9, 10\}$, $\{10, 4\}$, $\{4, 3\}$ and $\{3, 9\}$. By the arguments of (2), it is impossible.

For Case II, an exhausted search by hand shows that it is impossible to complete the partial design to a $(K_4 - e)$ -design of order 11.

This completes the proof. \square

Lemma 3.2 *Let (X, \mathcal{B}) be any $(K_4 - e)$ -design of order 11. Let $D = \{\{x, y\} : [x, y, z - u] \in \mathcal{B}\}$ and $N = \{\{z, u\} : [x, y, z - u] \in \mathcal{B}\}$. Then $D \cap N = \emptyset$.*

Proof Let $X = \{0, 1, 2, \dots, 10\}$. Assume that $D \cap N \neq \emptyset$. Without loss of generality let $\{\{0, 1\}\} \in D \cap N$ and

$$\mathcal{B}_0 = \{[0, 1, 2 - 3], [4, 5, 0 - 1], [0, 6, 7 - 8], [9, 10, 0 - x_1]\} \subset \mathcal{B}.$$

\mathcal{B}_0 is the set of all blocks containing 0.

Note that by the formula (1) in the proof of Lemma 3.1, for each $x \in X$ and each $i = 2, 3$, there are exactly 2 blocks in \mathcal{B} containing x in which the degree of x is i . This fact will be used OFTEN in the following!

Furthermore, consider the blocks containing 1. It is readily checked that up to

isomorphism, except for the blocks $[0, 1, 2 - 3]$ and $[4, 5, 0 - 1]$, the blocks containing 1 must be one of the following four cases:

- (1) $[6, 1, 9 - 10], [7, 8, 1 - x_2];$
- (2) $[7, 1, 8 - 9], [6, 10, 1 - x_2];$
- (3) $[9, 1, 6 - 7], [8, 10, 1 - x_2];$
- (4) $[9, 1, 7 - 8], [6, 10, 1 - x_2].$

Case (1). Let $\mathcal{B}_1 = \{[6, 1, 9 - 10], [7, 8, 1 - x_2]\}$. Then $\mathcal{B}_0 \cup \mathcal{B}_1 \subset \mathcal{B}$. It follows that $\{\{0, 1\}, \{1, 6\}, \{6, 0\}\} \subset D$, which is a cycle of length 3. By Lemma 3.1, a contradiction.

Case (2). Let $\mathcal{B}_1 = \{[7, 1, 8 - 9], [6, 10, 1 - x_2]\}$. Then $\mathcal{B}_0 \cup \mathcal{B}_1 \subset \mathcal{B}$. Consider the blocks containing 6, 7, 8, respectively. We have

$$\mathcal{B}_2 = \{[*], [*], [6 - 7], [*], [*], [6 - *], [*], [8, * - *], [*], [8, * - *], [*], [7, * - *]\} \subset \mathcal{B},$$

and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$. Since the edge $\{7, 10\}$ must occur in one block of \mathcal{B} , there must be a block of the form $[*, 7, 10 - *]$. Since the edge $\{8, 10\}$ must occur in one block of \mathcal{B} , there must be a block of the form $[*, 8, 10 - *]$. Since the edge $\{6, 9\}$ must occur in one block of \mathcal{B} , there must be a block of the form $[*, 9, 6 - *]$. Combining the above three facts, we rewrite \mathcal{B}_2 as follows:

$$\mathcal{B}_2 = \{[*], [*], [6 - 7], [*], [9, 6 - *], [*], [8, 10 - *], [*], [8, * - *], [*], [7, 10 - *]\}.$$

Consider the blocks containing the edge $\{8, 9\}$. We have the following two possibilities for \mathcal{B}_2 :

Subcase I. $[x_3, x_4, 6 - 7], [x_5, 9, 6 - x_6], [x_7, 8, 10 - 9], [x_8, 8, x_9, x_{10}], [x_{11}, 7, 10 - x_{12}];$

Subcase II. $[x_3, x_4, 6 - 7], [x_5, 9, 6 - x_6], [x_7, 8, 10 - x_8], [x_9, 8, 9, x_{10}], [x_{11}, 7, 10 - x_{12}].$

It is readily checked that for each $1 \leq i \leq 12$, $x_i \in \{2, 3, 4, 5\}$. Let $\{a, b, c, d\} = \{2, 3, 4, 5\}$. Consider the blocks containing the edges $\{2, 10\}, \{3, 10\}, \{4, 10\}, \{5, 10\}$. Then we can always take $x_1 = a$, $x_2 = b$, $x_7 = c$, $x_{11} = d$.

For Subcase I, consider the blocks containing 9. We have $x_5 = d$ and $x_6 = b$. Consider the blocks containing 6. We have $x_3 = a$ and $x_4 = c$. Consider the blocks containing 7. We have $x_{12} = b$. Thus $x_2 = x_6 = x_{12} = b$, and there are 3 blocks in \mathcal{B} containing b in which the degree of b is 2. A contradiction.

For Subcase II, consider the blocks containing the edge $\{6, a\}$. We have $x_3 = a$. Consider the block containing 7. We have $x_4 = c$ and $x_{12} = b$. Consider the block containing 6. We have $x_5 = d$. Consider the blocks containing 9. We have $x_6 = c$ and $x_9 = b$. Consider the blocks containing 8. We have $\{x_8, x_{10}\} = \{a, d\}$. If $x_8 = a$ and $x_{10} = d$, then the edge $\{a, c\}$ occurs in two blocks of \mathcal{B} . A contradiction. If $x_8 = d$ and $x_{10} = a$, then the edge $\{c, d\}$ occurs in two blocks of \mathcal{B} . A contradiction.

Case (3). Let $\mathcal{B}_1 = \{[9, 1, 6 - 7], [8, 10, 1 - x_2]\}$. Then $\mathcal{B}_0 \cup \mathcal{B}_1 \subset \mathcal{B}$. Consider the blocks containing 6, 7, 9, respectively. We have

$$\mathcal{B}_2 = \{[*], [7, * - *], [*], [7, * - *], [*], [6, * - *], [*], [*], [6 - 9], [*], [*], [9 - *]\} \subset \mathcal{B},$$

and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$. Since the edge $\{6, 10\}$ must occur in one block of \mathcal{B} , there must be a block of the form $[*, 6, 10 - *]$. Since the edge $\{7, 10\}$ must occur in one block of \mathcal{B} , there must be a block of the form $[*, 7, 10 - *]$. Since the edge $\{8, 9\}$ must occur in one block of \mathcal{B} , there must be a block of the form $[*, 8, 9 - *]$. Consider the blocks containing 8. Except the blocks $[0, 6, 7 - 8]$, $[8, 10, 1 - x_2]$ and $[*, 8, 9 - *]$, there must be a block of the form $[*, 7, 8 - *]$. Combining the above four facts, we have the following two possibilities for \mathcal{B}_2 :

Subcase I. $[x_3, 7, 10 - 8], [x_4, 7, x_5 - x_6], [x_7, 6, 10 - x_8], [x_9, x_{10}, 6 - 9], [x_{11}, 8, 9 - x_{12}]$;

Subcase II. $[x_3, 7, 10 - x_4], [x_5, 7, 8 - x_6], [x_7, 6, 10 - x_8], [x_9, x_{10}, 6 - 9], [x_{11}, 8, 9 - x_{12}]$.

It is readily checked that for each $1 \leq i \leq 12$, $x_i \in \{2, 3, 4, 5\}$. Let $\{a, b, c, d\} = \{2, 3, 4, 5\}$. Consider the blocks containing the edges $\{2, 10\}, \{3, 10\}, \{4, 10\}, \{5, 10\}$. Then we can always take $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_7 = d$. Consider the blocks containing 9. We have $x_9 = b$, $x_{10} = c$, $x_{11} = d$. Consider the blocks containing 6. We have $x_8 = a$.

For Subcase I, consider the blocks containing 8. We have $x_{12} = a$. Thus $x_1 = x_8 = x_{12} = a$, and there are 3 blocks in \mathcal{B} containing a in which the degree of a is 2. A contradiction.

For Subcase II, consider the blocks containing 8. We have $x_5 = a$ and $x_{12} = c$. Consider the blocks containing 7. We have $\{x_4, x_6\} = \{b, d\}$. If $x_4 = b$ and $x_6 = d$, then the edge $\{b, c\}$ occurs in two blocks. A contradiction. If $x_4 = d$ and $x_6 = b$, then the edge $\{c, d\}$ occurs in two blocks. A contradiction.

Case (4). Let $\mathcal{B}_1 = \{[9, 1, 7 - 8], [6, 10, 1 - x_2]\}$. Then $\mathcal{B}_0 \cup \mathcal{B}_1 \subset \mathcal{B}$. It follows that $\{\{0, 1\}, \{1, 9\}, \{9, 10\}, \{10, 6\}, \{6, 0\}\} \subset D$, which is a cycle of length 5. By Lemma 3.1, a contradiction.

This completes the proof. \square

Theorem 3.3 *There are exactly 2 non-isomorphic $(K_4 - e)$ -designs of order 11.*

Proof Let $X = \{0, 1, 2, \dots, 10\}$. Suppose that (X, \mathcal{B}) is any $(K_4 - e)$ -design of order 11. Let $D = \{\{x, y\} : [x, y, z - u] \in \mathcal{B}\}$ and $N = \{\{z, u\} : [x, y, z - u] \in \mathcal{B}\}$. By Lemma 3.1, (X, D) is a cycle of length 11. Without loss of generality, assume that $D = \{\{i, i+1\} : 0 \leq i \leq 9\} \cup \{\{10, 0\}\}$. Consider the blocks $[10, 0, a - b]$ and $[0, 1, c - d]$. We have $\{a, b\} \subset \{2, 3, 4, 5, 6, 7, 8\}$, $\{c, d\} \subset \{3, 4, 5, 6, 7, 8, 9\}$, and $\{a, b\} \cap \{c, d\} = \emptyset$. By Lemma 3.2, $D \cap N = \emptyset$. Due to $\{\{a, b\}, \{c, d\}\} \subset N$, it follows that $\{a, b\} \in \{\{j, k\} : 2 \leq j \leq 6, j+2 \leq k \leq 8\}$ and $\{c, d\} \in \{\{j, k\} : 3 \leq j \leq 7, j+2 \leq k \leq 9\}$.

By the formula (1) in the proof of Lemma 3.1, for every $x \in X$, there are exactly 2 blocks in \mathcal{B} containing x and satisfying the degree of x is 2. Let $[s_1, s_1 + 1, 0 - s_2]$ and $[t_1, t_1 + 1, 0 - t_2]$ be the two blocks containing 0 in which the degree of 0 is 2. Consider the edges containing 0. We have $\{10, 1, a, b, c, d, s_1, s_1 + 1, t_1, t_1 + 1\} = X \setminus \{0\}$. An exhaustive search by hand shows that there are 19 possibilities for the values a, b, c, d, s_1, t_1 satisfying

- (1) $\{a, b\} \in \{\{j, k\} : 2 \leq j \leq 6, j+2 \leq k \leq 8\};$
- (2) $\{c, d\} \in \{\{j, k\} : 3 \leq j \leq 7, j+2 \leq k \leq 9\};$
- (3) $\{10, 1, a, b, c, d, s_1, s_1 + 1, t_1, t_1 + 1\} = X \setminus \{0\}.$

We list them in the first five columns in Table I.

Table I

	$\{a, b\}$	$\{c, d\}$	$(s_1, s_1 + 1)$	$(t_1, t_1 + 1)$	$\pi\{a, b\}$	$\pi\{c, d\}$
1*	$\{2, 4\}$	$\{3, 5\}$	$(6, 7)$	$(8, 9)$	$\{7, 9\}$	$\{6, 8\}$
2*	$\{2, 4\}$	$\{3, 7\}$	$(5, 6)$	$(8, 9)$	$\{7, 9\}$	$\{4, 8\}$
3*	$\{2, 4\}$	$\{3, 9\}$	$(5, 6)$	$(7, 8)$	$\{7, 9\}$	$\{2, 8\}$
4*	$\{2, 5\}$	$\{6, 9\}$	$(3, 4)$	$(7, 8)$	$\{6, 9\}$	$\{2, 5\}$
5*	$\{2, 6\}$	$\{3, 7\}$	$(4, 5)$	$(8, 9)$	$\{5, 9\}$	$\{4, 8\}$
6*	$\{2, 6\}$	$\{3, 9\}$	$(4, 5)$	$(7, 8)$	$\{5, 9\}$	$\{2, 8\}$
7*	$\{2, 6\}$	$\{5, 7\}$	$(3, 4)$	$(8, 9)$	$\{5, 9\}$	$\{4, 6\}$
8*	$\{2, 6\}$	$\{5, 9\}$	$(3, 4)$	$(7, 8)$	$\{5, 9\}$	$\{2, 6\}$
9*	$\{2, 7\}$	$\{3, 6\}$	$(4, 5)$	$(8, 9)$	$\{4, 9\}$	$\{5, 8\}$
10*	$\{2, 8\}$	$\{3, 9\}$	$(4, 5)$	$(6, 7)$	$\{3, 9\}$	$\{2, 8\}$
11	$\{2, 8\}$	$\{7, 9\}$	$(3, 4)$	$(5, 6)$	$\{3, 9\}$	$\{2, 4\}$
12	$\{2, 8\}$	$\{5, 9\}$	$(3, 4)$	$(6, 7)$	$\{3, 9\}$	$\{2, 6\}$
13*	$\{4, 6\}$	$\{5, 7\}$	$(2, 3)$	$(8, 9)$	$\{5, 7\}$	$\{4, 6\}$
14	$\{4, 6\}$	$\{5, 9\}$	$(2, 3)$	$(7, 8)$	$\{5, 7\}$	$\{2, 6\}$
15	$\{4, 8\}$	$\{5, 9\}$	$(2, 3)$	$(6, 7)$	$\{3, 7\}$	$\{2, 6\}$
16	$\{4, 8\}$	$\{7, 9\}$	$(2, 3)$	$(5, 6)$	$\{3, 7\}$	$\{2, 4\}$
17*	$\{5, 7\}$	$\{4, 6\}$	$(2, 3)$	$(8, 9)$	$\{4, 6\}$	$\{5, 7\}$
18	$\{5, 8\}$	$\{4, 9\}$	$(2, 3)$	$(6, 7)$	$\{3, 6\}$	$\{2, 7\}$
19	$\{6, 8\}$	$\{7, 9\}$	$(2, 3)$	$(4, 5)$	$\{3, 5\}$	$\{2, 4\}$

Let $\pi = (0)(1\ 10)(2\ 9)(3\ 8)(4\ 7)(5\ 6)$ be a permutation on X . Obviously $\pi(D) = D$. Since $\pi([10, 0, a-b]) = [0, 1, \pi(a) - \pi(b)]$ and $\pi([0, 1, c-d]) = [10, 0, \pi(c) - \pi(d)]$, under the action of π , if for some possibility in Table I, whose values in the second and third column are $\{a, b\}$ and $\{c, d\}$, respectively, then it is isomorphic to the possibility with values $\pi\{c, d\}$ and $\pi\{a, b\}$ in the second and third column, respectively. Using this idea the above 19 possibilities can be reduced to 12 possibilities. We mark them with a *. Take the first possibility for example. In the first possibility, $\{a, b\} = \{2, 4\}$ and $\{c, d\} = \{3, 5\}$. Then $\pi\{c, d\} = \{6, 8\}$ and $\pi\{a, b\} = \{7, 9\}$, which corresponds to the last possibility.

Next consider the blocks $[0, 1, c-d]$ and $[1, 2, e-f]$. For each given $\{c, d\}$ in Table I, we need to determine all possible values of $\{e, f\}$. Fix the permutation $\sigma = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)$ on X and let A be the collection of the set $\{[10, 0, a-b], [0, 1, c-d]\}$, where $\{a, b\}, \{c, d\}$ are taken from the 19 possibilities in Table I. Let B be the collection of all possible cases of the set $\{[0, 1, c_1-d_1], [1, 2, e-f]\}$ such that one can complete a $(K_4 - e)$ -design originally from the two blocks $[0, 1, c_1-d_1]$ and $[1, 2, e-f]$. It is easy to see that $B \subseteq \sigma(A)$. Thus for determining B , we count $\sigma(A)$. Apply permutation σ to Table I to obtain Table II. Note that because

of $\{10, 1, a, b, c, d, s_1, s_1 + 1, t_1, t_1 + 1\} = X \setminus \{0\}$, we have $\sigma(\{10, 1, a, b, c, d, s_1, s_1 + 1, t_1, t_1 + 1\}) = X \setminus \{1\}$. Here $[s'_1, s'_1 + 1, 1 - s'_2]$ and $[t'_1, t'_1 + 1, 1 - t'_2]$ are the two blocks containing 1 in which the degree of 1 is 2.

Table II

	$\{c_1, d_1\}$	$\{e, f\}$	$(s'_1, s'_1 + 1)$	$(t'_1, t'_1 + 1)$
1	$\{3, 5\}$	$\{4, 6\}$	$(7, 8)$	$(9, 10)$
2	$\{3, 5\}$	$\{4, 8\}$	$(6, 7)$	$(9, 10)$
3	$\{3, 5\}$	$\{4, 10\}$	$(6, 7)$	$(8, 9)$
4	$\{3, 6\}$	$\{7, 10\}$	$(4, 5)$	$(8, 9)$
5	$\{3, 7\}$	$\{4, 8\}$	$(5, 6)$	$(9, 10)$
6	$\{3, 7\}$	$\{4, 10\}$	$(5, 6)$	$(8, 9)$
7	$\{3, 7\}$	$\{6, 8\}$	$(4, 5)$	$(9, 10)$
8	$\{3, 7\}$	$\{6, 10\}$	$(4, 5)$	$(8, 9)$
9	$\{3, 8\}$	$\{4, 7\}$	$(5, 6)$	$(9, 10)$
10	$\{3, 9\}$	$\{4, 10\}$	$(5, 6)$	$(7, 8)$
11	$\{3, 9\}$	$\{8, 10\}$	$(4, 5)$	$(6, 7)$
12	$\{3, 9\}$	$\{6, 10\}$	$(4, 5)$	$(7, 8)$
13	$\{5, 7\}$	$\{6, 8\}$	$(3, 4)$	$(9, 10)$
14	$\{5, 7\}$	$\{6, 10\}$	$(3, 4)$	$(8, 9)$
15	$\{5, 9\}$	$\{6, 10\}$	$(3, 4)$	$(7, 8)$
16	$\{5, 9\}$	$\{8, 10\}$	$(3, 4)$	$(6, 7)$
17	$\{6, 8\}$	$\{5, 7\}$	$(3, 4)$	$(9, 10)$
18	$\{6, 9\}$	$\{5, 10\}$	$(3, 4)$	$(7, 8)$
19	$\{7, 9\}$	$\{8, 10\}$	$(3, 4)$	$(5, 6)$

Now for each given $\{a, b\}$ and $\{c, d\}$ in Table I, we can use Table II to determine all possible values of $\{e, f\}$. For example when we take the first possibility in Table I, i.e., $\{a, b\} = \{2, 4\}$ and $\{c, d\} = \{3, 5\}$, the values of (e, f) can be taken from the rows with $(c_1, d_1) = (3, 5)$ in Table II. Thus we have $(e, f) = (4, 6)$, $(4, 8)$ or $(4, 10)$. These three subcases corresponding to the first possibility in Table I are listed below.

Table III

	$\{a, b\}$	$\{c, d\}$	$\{e, f\}$	$(s_1, s_1 + 1)$	$(t_1, t_1 + 1)$	$(s'_1, s'_1 + 1)$	$(t'_1, t'_1 + 1)$
1*	$\{2, 4\}$	$\{3, 5\}$	$\{4, 6\}$	$(6, 7)$	$(8, 9)$	$(7, 8)$	$(9, 10)$
			$\{4, 8\}$			$(6, 7)$	$(9, 10)$
			$\{4, 10\}$			$(6, 7)$	$(8, 9)$

For reducing these subcases, we notice that s_1 can not be equal to s'_1 . Otherwise there would be a block $[s_1, s_1 + 1, 0 - 1] \in \mathcal{B}$, which implies $\{0, 1\} \in N$. Due to $\{0, 1\} \in D$ and $D \cap N = \emptyset$ from Lemma 3.2, a contradiction occurs. Similarly, we have $s_1 \neq t'_1$, $t_1 \neq s'_1$ and $t_1 \neq t'_1$. Thus $|\{s_1, t_1, s'_1, t'_1\}| = 4$. Using this condition, for each given $\{a, b\}$ and $\{c, d\}$ in Table I, we can reduce possible values of $\{e, f\}$. For example in Table III only the first subcase satisfies $|\{s_1, t_1, s'_1, t'_1\}| = 4$. After exhaustive search by hand, we can reduce the 12 possibilities marked * in Table I to 6 possibilities in Table IV.

Table IV

	$\{a, b\}$	$\{c, d\}$	$\{e, f\}$	$(s_1, s_1 + 1)$	$(t_1, t_1 + 1)$	$(s'_1, s'_1 + 1)$	$(t'_1, t'_1 + 1)$
1*	$\{2, 4\}$	$\{3, 5\}$	$\{4, 6\}$	$(6, 7)$	$(8, 9)$	$(7, 8)$	$(9, 10)$
2*	$\{2, 4\}$	$\{3, 7\}$	$\{6, 8\}$	$(5, 6)$	$(8, 9)$	$(4, 5)$	$(9, 10)$
3*	$\{2, 4\}$	$\{3, 9\}$	$\{8, 10\}$	$(5, 6)$	$(7, 8)$	$(4, 5)$	$(6, 7)$
5*	$\{2, 6\}$	$\{3, 7\}$	$\{4, 8\}$	$(4, 5)$	$(8, 9)$	$(5, 6)$	$(9, 10)$
10*	$\{2, 8\}$	$\{3, 9\}$	$\{4, 10\}$	$(4, 5)$	$(6, 7)$	$(5, 6)$	$(7, 8)$
13*	$\{4, 6\}$	$\{5, 7\}$	$\{6, 8\}$	$(2, 3)$	$(8, 9)$	$(3, 4)$	$(9, 10)$

In the following we show that Possibilities 2, 3, 10 and 13 in Table IV are impossible. For Possibilities 2 and 3, consider the block containing the edge $\{2, 4\}$. It must be of the form $[2, 3, 4 - *]$ (note that $N \cap D = \emptyset$). Since there are four blocks containing 2 in \mathcal{B} , and it is easy to verify that one can not find the fourth block containing 2, a contradiction occurs. For Possibility 10, the edge $\{2, 10\}$ occurs in two blocks $[10, 0, 2 - 8]$ and $[1, 2, 4 - 10]$. A contradiction. For Possibility 13, consider the block containing the edge $\{4, 6\}$. It must be of the form $[6, 7, 4 - *]$. Consider the block containing the edge $\{6, 8\}$. It must be of the form $[5, 6, 8 - *]$. Then all blocks containing 4 are of the form $[10, 0, 4 - 6]$, $[6, 7, 4 - *]$, $[3, 4, * - *]$, $[4, 5, * - *]$, and all blocks containing 8 are of the form $[1, 2, 6 - 8]$, $[5, 6, 8 - *]$, $[7, 8, * - *]$, $[8, 9, * - *]$. It follows that there is no block containing the edge $\{4, 8\}$. A contradiction.

By Possibilities 1 and 5 in Table IV, the blocks $[10, 0, a - b]$ and $[0, 1, c - d]$ must be one of the following two cases: (1) $[10, 0, 2 - 4]$, $[0, 1, 3 - 5]$; (2) $[10, 0, 2 - 6]$, $[0, 1, 3 - 7]$. It implies that for any $i \in X$, the blocks $[i, i + 1, x - y]$ and $[i + 1, i + 2, z - u]$ must be one of the following two cases: (1) $[i, i + 1, (i + 3) - (i + 5)]$, $[i + 1, i + 2, (i + 4) - (i + 6)]$; (2) $[i, i + 1, (i + 3) - (i + 7)]$, $[i + 1, i + 2, (i + 4) - (i + 8)]$, where the arithmetic is modulo 11. It follows that \mathcal{B} must be one of the following two cases:

$$\begin{aligned}\mathcal{B}_1 &= \{[i, i + 1, (i + 3) - (i + 5)] : i \in X\}, \\ \mathcal{B}_2 &= \{[i, i + 1, (i + 3) - (i + 7)] : i \in X\}.\end{aligned}$$

It is readily checked that (X, \mathcal{B}_1) and (X, \mathcal{B}_2) are both $(K_4 - e)$ -designs, and they are non-isomorphic. This completes the proof. \square

4 Application in fine triangle intersection problem

As an application of our enumerative results, in this section we investigate the fine triangle intersection problem for $(K_4 - e)$ -designs of orders $v = 6, 10, 11$.

Let B be a simple graph. Denote by $T(B)$ the set of all triangles of the graph B . For example, if $B = [a, b, c - d]$, then $T(B) = \{\{a, b, c\}, \{a, b, d\}\}$. Two G -designs of order v (X, \mathcal{B}_1) and (X, \mathcal{B}_2) intersect in t triangles provided $|T(\mathcal{B}_1) \cap T(\mathcal{B}_2)| = t$, where $T(\mathcal{B}_i) = \bigcup_{B \in \mathcal{B}_i} T(B)$, $i = 1, 2$. Define $Fin_G(v) = \{(s, t) : \exists \text{ a pair of } G\text{-designs of order } v \text{ intersecting in } s \text{ blocks and } t + s|T(G)| \text{ triangles}\}$. The *fine triangle intersection problem* for G -designs is to determine $Fin_G(v)$.

The fine triangle intersection problem for G -designs, which was introduced in [7], is the generalization of the intersection problem and the triangle intersection problem for G -designs. For more information on the intersection problem for G -designs, the interested reader may refer to [3, 4, 8, 9, 10, 11]. For more information on the triangle intersection problem for G -designs, the interested reader may refer to [5, 6, 12].

Let $b_v = v(v-1)/10$ be the number of blocks in a $(K_4 - e)$ -design, and $[a, b]$ the set of all integers x satisfying $a \leq x \leq b$. Let $J(v) = \{s : \text{there exists a pair of } (K_4 - e)\text{-designs of order } v \text{ intersecting in } s \text{ blocks}\}$, and $J_T(v) = \{t : \text{there exists a pair of } (K_4 - e)\text{-designs of order } v \text{ intersecting in } t \text{ triangles}\}$.

Theorem 4.1 ([3]) *For any $v \equiv 0, 1 \pmod{5}$, $v \geq 6$ and $v \neq 11$, $J(v) = [0, b_v] \setminus \{b_v - 1, b_v - 2\}$; $J(11) = \{0, 1, 2, \dots, 6, 11\}$.*

Theorem 4.2 ([5]) *For any $v \equiv 0, 1 \pmod{5}$, $v \geq 15$, $J_T(v) = [0, 2b_v] \setminus \{2b_v - 1, 2b_v - 2\}$; $J_T(6) = \{0, 2, 3, 6\}$; $J_T(10) = \{0, 1, \dots, 12, 14, 15, 18\}$; $J_T(11) = \{0, 1, \dots, 16, 22\}$.*

If a pair of $(K_4 - e)$ -designs have blocks in common, each common block contributes 2 common triangles. In what follows we always write $Fin_G(v)$ simply as $Fin(v)$ when G is the graph $K_4 - e$, i.e., $Fin(v) = \{(s, t) : \exists \text{ a pair of } (K_4 - e)\text{-designs of order } v \text{ intersecting in } s \text{ blocks and } t + 2s \text{ triangles}\}$. Let $Adm(v) = \{(s, t) : s + t \leq b_v, s \in J(v), t + 2s \in J_T(v)\}$. From the definitions of $Fin(v)$, $J(v)$ and $J_T(v)$, it is clear that $Fin(v) \subseteq Adm(v)$.

Theorem 4.3 $Fin(6) = Adm(6)$.

Proof Let $X = \{0, 1, 2, 3, 4, 5\}$ and $\mathcal{B} = \{[0, 1, 2 - 3], [2, 3, 4 - 5], [4, 5, 0 - 1]\}$. Then (X, \mathcal{B}) is a $(K_4 - e)$ -design of order 6. Consider the following permutations on X .

$$\pi_{0,0} = (2\ 4)(3\ 5), \quad \pi_{0,2} = (1\ 2), \quad \pi_{0,3} = (1\ 3)(2\ 4), \quad \pi_{3,0} = (1).$$

Then we have $|\pi_{s,t}\mathcal{B} \cap \mathcal{B}| = s$ and $|T(\pi_{s,t}\mathcal{B} \setminus \mathcal{B}) \cap T(\mathcal{B} \setminus \pi_{s,t}\mathcal{B})| = t$ for each $(s, t) \in Adm(6)$. \square

Theorem 4.4 $Fin(10) = Adm(10) \setminus \{(1, 8), (3, 1), (3, 5), (4, 1), (4, 3), (5, 1), (5, 2)\}$.

Proof Take the same $(K_4 - e)$ -designs of order 10 (X, \mathcal{B}_i) , $i = 1, 2, 3$, as those in the proof of Theorem 2.3, which are mutually non-isomorphic. Consider the following permutations on X .

$$\begin{array}{llll} \pi_{0,0} = (2\ 4)(3\ 5)(7\ 8), & \pi_{0,1} = (3\ 4\ 6)(5\ 8\ 9), & \pi_{0,2} = (3\ 4\ 5\ 6)(7\ 8), & \pi_{0,3} = (3\ 4)(5\ 6)(7\ 8), \\ \pi_{0,4} = (3\ 4)(7\ 8), & \pi_{0,5} = (3\ 6)(5\ 8), & \pi_{0,6} = (1\ 2)(4\ 6)(5\ 7), & \pi_{0,7} = (1\ 2)(3\ 4), \\ \pi_{0,8} = (1\ 2)(3\ 6)(5\ 7), & \pi_{0,9} = (1\ 2)(3\ 4)(8\ 9), & \pi_{1,0} = (4\ 6)(5\ 8\ 9), & \pi_{1,1} = (5\ 6)(7\ 8\ 9), \\ \pi_{1,2} = (2\ 4)(3\ 6), & \pi_{1,3} = (5\ 6)(7\ 8), & \pi_{1,4} = (3\ 4)(5\ 6), & \pi_{1,5} = (3\ 6)(5\ 7), \\ \pi_{1,6} = (1\ 6)(2\ 4\ 5\ 9)(3\ 7), & \pi_{1,7} = (1\ 6)(5\ 8), & \pi_{2,0} = (4\ 6)(5\ 7\ 8\ 9), & \pi_{2,1} = (5\ 6)(8\ 9), \\ \pi_{2,2} = (4\ 6)(5\ 7\ 8), & \pi_{2,3} = (3\ 4\ 6)(5\ 7), & \pi_{2,4} = (3\ 6)(4\ 8)(5\ 9), & \pi_{2,5} = (3\ 5\ 7\ 4\ 6)(8\ 9), \\ \pi_{2,6} = (3\ 4), & \pi_{2,7} = (1\ 2)(3\ 5), & \pi_{3,0} = (7\ 8\ 9), & \pi_{3,2} = (4\ 5)(7\ 8), \\ \pi_{3,3} = (5\ 6), & \pi_{3,4} = (0\ 2\ 4)(1\ 3\ 5), & \pi_{3,6} = (0\ 2\ 4)(1\ 3\ 5)(7\ 8), & \pi_{4,0} = (4\ 6)(5\ 7)(8\ 9), \\ \pi_{4,2} = (4\ 6)(5\ 7), & \pi_{4,4} = (4\ 6)(5\ 7), & \pi_{5,0} = (8\ 9), & \pi_{5,4} = (4\ 5), \\ \pi_{6,0} = (2\ 4)(3\ 5)(8\ 9), & \pi_{6,2} = (3\ 5), & \pi_{6,3} = (0\ 2)(1\ 4)(7\ 8), & \pi_{9,0} = (1). \end{array}$$

Let $E = \{(1, 8), (3, 1), (3, 5), (4, 1), (4, 3), (5, 1), (5, 2)\}$ and $M = \{(1, 6), (1, 7), (2, 5), (2, 7), (3, 2), (3, 4), (3, 6), (4, 0), (4, 4), (5, 4), (6, 2), (6, 3)\}$. Then for each $(s, t) \in \text{Adm}(10) \setminus (E \cup M)$, $|\pi_{s,t}\mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $|T(\pi_{s,t}\mathcal{B}_1 \setminus \mathcal{B}_1) \cap T(\mathcal{B}_1 \setminus \pi_{s,t}\mathcal{B}_1)| = t$. For each $(s, t) \in M$, $|\pi_{s,t}\mathcal{B}_2 \cap \mathcal{B}_2| = s$ and $|T(\pi_{s,t}\mathcal{B}_2 \setminus \mathcal{B}_2) \cap T(\mathcal{B}_2 \setminus \pi_{s,t}\mathcal{B}_2)| = t$.

Now it remains to show that for each $(s, t) \in E$, we have $(s, t) \notin \text{Fin}(10)$. By Theorem 2.3, there are exactly 3 non-isomorphic $(K_4 - e)$ -designs of order 10. Thus we can check all the cases by computer exhaustive search for the fine triangle intersection numbers of a pair of $(K_4 - e)$ -designs of order 10, i.e., for any permutation π on X and for each $i, j = 1, 2, 3$, count $|\pi\mathcal{B}_i \cap \mathcal{B}_j|$ and $|T(\pi\mathcal{B}_i \setminus \mathcal{B}_j) \cap T(\mathcal{B}_j \setminus \pi\mathcal{B}_i)|$. This completes the proof. \square

Theorem 4.5 $\text{Fin}(11) = \text{Adm}(11) \setminus \{(3, 0), (4, 0), (4, 6), (5, 0), (5, 1), (5, 2), (6, 0), (6, 1), (6, 3), (6, 4)\}$.

Proof Take $X = \{0, 1, 2, \dots, 10\}$ and $\mathcal{B}_1 = \{[i, i + 1, (i + 3) - (i + 5)] : i \in X\}$, $\mathcal{B}_2 = \{[i, i + 1, (i + 3) - (i + 7)] : i \in X\}$, where the arithmetic is modulo 11. By Theorem 3.3, (X, \mathcal{B}_1) and (X, \mathcal{B}_2) are two non-isomorphic $(K_4 - e)$ -designs of order 11. Consider the following permutations on X .

$$\begin{array}{lll}
\pi_{0,0} = (0\ 8\ 5\ 4\ 6)(2\ 9\ 10\ 3), & \pi_{0,1} = (0\ 8\ 1\ 7)(2\ 3\ 9\ 6\ 5\ 4), & \pi_{0,2} = (0\ 10\ 8\ 4)(2\ 9\ 6\ 7\ 3), \\
\pi_{0,3} = (0\ 3)(2\ 8\ 7\ 4\ 5\ 10\ 6\ 9), & \pi_{0,4} = (2\ 6\ 8\ 3\ 5\ 9\ 10\ 7\ 4), & \pi_{0,5} = (0\ 9\ 4)(1\ 6\ 10\ 3)(2\ 5\ 8\ 7), \\
\pi_{0,6} = (0\ 5\ 10\ 9)(1\ 7\ 2)(3\ 8), & \pi_{0,7} = (0\ 2\ 3\ 9\ 6)(1\ 8)(4\ 10), & \pi_{0,8} = (0\ 3\ 2\ 6\ 8)(1\ 5\ 4\ 9\ 7), \\
\pi_{0,9} = (0\ 3\ 7\ 6\ 8\ 2)(1\ 10)(5\ 9), & \pi_{0,10} = (0\ 7\ 10\ 6\ 8\ 5\ 2\ 1\ 3\ 9\ 4), & \pi_{0,11} = (0\ 2\ 6\ 1\ 7\ 4)(5\ 9\ 8), \\
\pi_{1,0} = (0\ 8\ 3\ 4\ 10)(1\ 2\ 5\ 7\ 9\ 6), & \pi_{1,1} = (0\ 2\ 7\ 3\ 8)(1\ 6\ 4\ 10\ 5\ 9), & \pi_{1,2} = (0\ 10\ 2\ 1\ 5)(3\ 7\ 4\ 6\ 9), \\
\pi_{1,3} = (0\ 2\ 6\ 7\ 8\ 9)(1\ 4\ 3\ 5\ 10), & \pi_{1,4} = (0\ 1\ 7\ 5\ 10\ 9\ 6\ 3)(4\ 8), & \pi_{1,5} = (0\ 5\ 8\ 7)(1\ 2\ 4\ 6)(3\ 10\ 9), \\
\pi_{1,6} = (0\ 3\ 7\ 1\ 8\ 10\ 5\ 4\ 2\ 9\ 6), & \pi_{1,7} = (0\ 5\ 3\ 10)(1\ 2\ 7\ 8\ 9\ 4), & \pi_{1,8} = (0\ 8\ 10\ 7\ 5\ 3)(1\ 6\ 4\ 2), \\
\pi_{1,9} = (0\ 10\ 2\ 8\ 9\ 3\ 7\ 4)(1\ 5), & \pi_{1,10} = (1\ 10\ 3\ 4\ 9\ 6\ 5\ 2)(7\ 8), & \pi_{2,0} = (0\ 10\ 5\ 6)(1\ 7)(2\ 3\ 9\ 4\ 8), \\
\pi_{2,1} = (0\ 8\ 1\ 7\ 6\ 3\ 9\ 5\ 2)(4\ 10), & \pi_{2,2} = (0\ 2\ 3\ 6\ 10\ 1)(4\ 5)(7\ 9), & \pi_{2,3} = (0\ 1\ 5\ 9\ 7\ 3\ 10\ 2\ 6\ 8), \\
\pi_{2,4} = (1\ 8\ 5)(2\ 10\ 4)(6\ 7), & \pi_{2,5} = (0\ 10\ 4\ 5\ 9\ 3\ 8\ 1\ 6\ 7\ 2), & \pi_{2,6} = (0\ 7\ 3)(1\ 6\ 10\ 8\ 4\ 2)(5\ 9), \\
\pi_{2,7} = (1\ 3\ 6\ 7\ 4\ 5), & \pi_{2,8} = (0\ 2\ 4\ 8)(1\ 3\ 7\ 6)(9\ 10), & \pi_{2,9} = (0\ 4\ 6)(1\ 9\ 2\ 8)(3\ 7\ 10), \\
\pi_{3,1} = (0\ 9\ 7\ 5\ 3)(1\ 2)(4\ 6)(8\ 10), & \pi_{3,2} = (0\ 5\ 1\ 9\ 3\ 7\ 10\ 4)(2\ 8\ 6), & \pi_{3,3} = (0\ 7\ 3\ 10\ 6\ 4)(1\ 8\ 5)(2\ 9), \\
\pi_{3,4} = (0\ 9\ 5)(1\ 8\ 2\ 7\ 3\ 6\ 10\ 4), & \pi_{3,5} = (0\ 10\ 4\ 2)(1\ 9\ 7\ 5\ 3)(6\ 8), & \pi_{3,6} = (1\ 10\ 8\ 6\ 9\ 7\ 5\ 2)(3\ 4), \\
\pi_{3,7} = (0\ 2)(3\ 6), & \pi_{3,8} = (0\ 3\ 8\ 2\ 7\ 1\ 6)(4\ 9\ 5), & \pi_{4,1} = (0\ 2\ 6\ 5\ 10\ 3\ 1\ 7\ 4)(8\ 9), \\
\pi_{4,2} = (0\ 8\ 9\ 4\ 5\ 6\ 7\ 10\ 3\ 1), & \pi_{4,3} = (0\ 5\ 9\ 1\ 4\ 8)(2\ 3\ 7\ 10), & \pi_{4,4} = (0\ 6\ 9\ 2\ 4\ 8\ 1\ 5\ 10\ 3\ 7), \\
\pi_{4,5} = (0\ 3\ 6)(1\ 4\ 7\ 8\ 9)(2\ 5\ 10), & \pi_{4,7} = (0\ 1\ 5\ 9\ 7\ 3)(2\ 6)(4\ 10), & \pi_{5,3} = (0\ 8\ 7\ 3\ 1\ 10\ 9\ 6\ 4\ 2), \\
\pi_{5,4} = (0\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 1), & \pi_{5,5} = (0\ 10)(1\ 3\ 9\ 5)(2\ 4\ 8\ 6), & \pi_{5,6} = (0\ 4\ 7\ 1\ 3\ 6\ 9\ 10\ 5\ 8), \\
\pi_{6,2} = (0\ 7\ 3\ 10\ 8\ 4)(1\ 6\ 2\ 9\ 5), & \pi_{11,0} = (1). &
\end{array}$$

Let $E = \{(3, 0), (4, 0), (4, 6), (5, 0), (5, 1), (5, 2), (6, 0), (6, 1), (6, 3), (6, 4)\}$ and $M = \{(4, 1), (4, 2), (4, 7), (5, 5)\}$. Then for each $(s, t) \in \text{Adm}(10) \setminus (E \cup M)$, $|\pi_{s,t}\mathcal{B}_1 \cap \mathcal{B}_1| = s$ and $|T(\pi_{s,t}\mathcal{B}_1 \setminus \mathcal{B}_1) \cap T(\mathcal{B}_1 \setminus \pi_{s,t}\mathcal{B}_1)| = t$. For each $(s, t) \in M$, $|\pi_{s,t}\mathcal{B}_2 \cap \mathcal{B}_1| = s$ and $|T(\pi_{s,t}\mathcal{B}_2 \setminus \mathcal{B}_1) \cap T(\mathcal{B}_1 \setminus \pi_{s,t}\mathcal{B}_2)| = t$.

Now it remains to show that for each $(s, t) \in E$, we have $(s, t) \notin \text{Fin}(11)$. By Theorem 3.3, there are exactly 2 non-isomorphic $(K_4 - e)$ -designs of order 11. Thus we can check all the cases by computer exhaustive search for the fine triangle intersection numbers of a pair of $(K_4 - e)$ -designs of order 11, i.e., for any permutation π on X and for each $i, j = 1, 2$, count $|\pi\mathcal{B}_i \cap \mathcal{B}_j|$ and $|T(\pi\mathcal{B}_i \setminus \mathcal{B}_j) \cap T(\mathcal{B}_j \setminus \pi\mathcal{B}_i)|$. This completes the proof. \square

Remark: In this paper, we focus on the enumerations of $(K_4 - e)$ -designs of orders $v = 6, 10, 11$. As an application the fine triangle intersection problem for $(K_4 - e)$ -designs of orders $v = 6, 10, 11$ are considered. The determination of the set $Fin(v)$ is currently being investigated for any $v \equiv 0, 1 \pmod{5}$ and $v \geq 6$.

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